

Integer properties

Sign of a power with a negative base
If n is even, $(-x)^n > 0$ and if n is odd,
 $(-x)^n < 0$

The relationship between dividend, divisor,
quotient, and remainder
 $n = d \times q + r$

Digit tests
 $d = d_1 \cdot d_2, \gcd(d_1, d_2) = 1 \Rightarrow d \mid n$
 $\Leftrightarrow d_1 \mid n \text{ and } d_2 \mid n$

Every integer greater than 1 is a unique
product of prime powers
 $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

Number of positive factors of n
 $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$

The GCD and LCM of two positive integers
multiply to the product of the two integers.
 $\gcd(a, b) \times \text{lcm}(a, b) = a \times b$

The exponent of a prime p in $n!$ is
 $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$

The product of two consecutive integers is
always even.
 $n(n+1) = n^2 + n$

Fractions and the number line

A fraction reduced to lowest terms by dividing
numerator and denominator by their common
factor

$$\frac{a}{b} = \frac{a \div g}{b \div g}, g = \gcd(a, b)$$

Sum or difference of two fractions over their
least common denominator

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot \frac{L}{b} \pm c \cdot \frac{L}{d}}{L},$$

$L = \text{least common denominator}$

Fraction comparison for positive
denominators

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc$$

Multiplying and dividing fractions
 $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ and $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

Remaining amount after successive fractions
taken from what is left
 $L = A \cdot (1 - f_1)(1 - f_2) \cdots (1 - f_n)$

Original amount recovered from a final
leftover

$$O = \frac{L}{(1 - f_1)(1 - f_2) \cdots (1 - f_n)}$$

For positive denominators, $a/b > c/d$ exactly
when $ad > bc$.

$$\frac{a}{b} > \frac{c}{d} \text{ when } ad > bc \ (b > 0, d > 0)$$

A decimal with k digits after the point
becomes a fraction with those digits on top
and 10^k on the bottom.

$$0.d_1 d_2 \cdots d_k = \frac{d_1 d_2 \cdots d_k}{10^k}$$

The position of the n th digit within the repeat
cycle
 $(n - k) \bmod L$

A two-place decimal with unknown digits a
and b

$$0.ab = \frac{10a + b}{100}$$

The sign-revealing identities

$$|x| = -x \text{ when } x \leq 0, |x| = x$$

when $x \geq 0, \sqrt{x^2} = |x|$

The words "x is within b units of a" become a
single interval.

$$|x - a| < b \text{ becomes } a - b < x < a + b$$

Exponents and roots

For powers with the same base, combine
exponents by adding for multiplication and
subtracting for division.

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

Raise a power to another power by multiplying
the exponents.
 $(a^m)^n = a^{mn}$

Power of a product
 $(ab)^n = a^n b^n$

A composite base raised to a power can be
rewritten using its prime factor.
 $b^m = (p^k)^m = p^{km}$

Factoring a sum of two powers with the same
base

$$a^m + a^n = a^m(1 + a^{n-m}), m \leq n$$

The value of a base raised to the zero or a
negative power

$$a^0 = 1, a^{-n} = \frac{1}{a^n}, \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

A base raised to a fractional exponent m over
 n

$$a^{m/n} = (\sqrt[n]{a})^m$$

The square root of a product equals the
product of the square roots, for nonnegative
values.

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

The product of two square roots equals the
square root of the product.

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

Rationalizing a binomial denominator by
multiplying by the conjugate

$$\frac{1}{a + \sqrt{b}} \cdot \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$$

Signed roots

$$x^2 = n \Rightarrow x = \pm \sqrt{n}, \sqrt{n} \geq 0,$$

$$\sqrt{x^2} = |x|$$

The cube root of a negative number is the
negative of the cube root of the positive
version of that number.

$$\sqrt[3]{-a} = -\sqrt[3]{a}$$

If a radicand lies between consecutive perfect
squares, its square root lies between those
squares' roots

$$p^2 < n < (p+1)^2 \Rightarrow p < \sqrt{n} < p+1$$

When two powers of the same base are equal,
set the exponents equal.

$$a^x = a^y \Rightarrow x = y \text{ (for } a \neq 0, 1, -1)$$

Rewrite each side as a power of a common
base, then equate exponents.

$$(p^k)^x = p^j \Rightarrow kx = j$$

Algebra and inequalities

Combining like terms by adding or subtracting
their coefficients

$$ax^n + bx^n = (a + b)x^n$$

Distributing a factor across parentheses
multiplies every inside term by that factor.

$$a(b + c) = ab + ac$$

Factoring out the greatest common factor
 $ab + ac = a(b + c)$

Adding or subtracting two fractions with
different denominators

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

The three special product identities

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Difference of fourth powers, factored
recursively

$$x^4 - y^4 = (x^2 + y^2)(x + y)(x - y)$$

The product of two numbers near a round
base, and the square of a number near a round
base

$$(n + k)(n - k) = n^2 - k^2$$

$$(n \pm k)^2 = n^2 \pm 2nk + k^2$$

The sum of the squares of two variables, given
their sum and product

$$a^2 + b^2 = (a + b)^2 - 2ab$$

The product of two radical conjugates

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$$

The solution to a linear equation $ax + b = c$,
with a not 0

$$x = \frac{c - b}{a}$$

For an equation with the variable on both
sides, collect all variable terms on one side
before isolating.

$$ax + b = cx + d \Rightarrow (a - c)x = d - b$$

$$\Rightarrow x = \frac{d - b}{a - c} \text{ when } a \neq c$$

Distribute a multiplier across each term inside
parentheses, carrying any negative sign with
it

$$a(bx + c) = abx + ac$$

Target combinations

$$ax + ay = c \Rightarrow a(x + y) = c$$

$$\Rightarrow x + y = \frac{c}{a}$$

Linear inequalities

$$ax + b \leq c \Rightarrow x \leq \frac{c - b}{a} \text{ if } a > 0;$$

$$x \geq \frac{c - b}{a} \text{ if } a < 0$$

Compound inequalities

$$a < bx + c < d$$

Range of a product of two bounded variables

$$\min(ac, ad, bc, bd) \leq xy \leq \max(ac, ad, bc, bd)$$

when $a \leq x \leq b, c \leq y \leq d$

Quadratics

Sum and product of the roots of a quadratic

$$r + s = -\frac{b}{a} \text{ and } rs = \frac{c}{a}$$

Standard form for factoring a quadratic

$$ax^2 + bx + c = 0$$

The difference of two squares factors into the
product of the sum and difference of their
square roots.

$$x^2 - k^2 = (x - k)(x + k)$$

The roots of $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The sum and product of the roots of $ax^2 + bx + c = 0$

$$r + s = -\frac{b}{a}, rs = \frac{c}{a}$$

The sum of the squares of the roots, in terms
of the sum and product

$$r^2 + s^2 = (r + s)^2 - 2rs$$

Clearing a denominator from a relation with a
variable in the denominator

$$x + \frac{a}{x} = b \Rightarrow x^2 - bx + a = 0$$

Product of two quantities with a known
difference k

$$n(n + k) = P$$

Total revenue is the product of price and
quantity.

$$R = p \times q$$

Discriminant

$$\Delta = b^2 - 4ac$$

$\Delta > 0$: two real roots

$\Delta = 0$: one real root

$\Delta < 0$: no real roots

Functions and sequences

Evaluating a two-input custom operation

$$x \# y = R(x, y)$$

A recursive rule builds each term from the one
before it.

$$a_{n+1} = a_n + d \text{ or } a_{n+1} = r \cdot a_n$$

The n th term of an arithmetic sequence

$$a_n = a_1 + (n - 1)d$$

The common difference from any two terms

$$d = \frac{a_j - a_i}{j - i}$$

The n th term of a geometric sequence from its
first term and common ratio

$$a_n = a_1 r^{n-1}$$

Position of a given value in an arithmetic
sequence

$$n = 1 + \frac{a_n - a_1}{d}$$

The sum of an arithmetic sequence equals the
number of terms times the average of the first
and last terms.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Percents and interest

Percent change, as a percent of the original
value

$$\text{percent change} = \frac{\text{new} - \text{original}}{\text{original}} \times 100$$

Recover the original by dividing the new value
by the multiplier

$$O = \frac{N}{m}$$

List price from cost with a markup percent,
and selling price from list price with a discount
percent

$$L = C \left(1 + \frac{m}{100}\right) \text{ and}$$

$$S = L \left(1 - \frac{d}{100}\right)$$

Net multiplier for two successive changes

$$M_{\text{net}} = \left(1 + \frac{p_1}{100}\right) \left(1 + \frac{p_2}{100}\right)$$

A quantity A that is p percent more than a
base B

$$A = B \left(1 + \frac{p}{100}\right)$$

If A is $p\%$ more than B , then B is $q\%$ less than
 A , where

$$q = \frac{100p}{100 + p}$$

The combined multiplier for a chain of two
comparisons

$$M_{A,C} = M_{A,B} \times M_{B,C}$$

When p percent of x equals q percent of y

$$\frac{p}{100} \cdot x = \frac{q}{100} \cdot y$$

Simple interest adds a fixed amount each
period, while compound interest multiplies
the balance each period.

$$I = Prt \text{ and } B = P \left(1 + \frac{r}{n}\right)^{nt}$$

The gap between compound and simple
balance after two years, same principal and
rate

$$P(1 + r)^2 - P(1 + 2r) = Pr^2$$

Rate from two consecutive balances

$$r = \frac{B_2}{B_1} - 1$$

Ratios, rates and work

Each part's fraction of the whole, from ratio a:b

a/(a+b) and b/(a+b)

When a ratio a:b gives amounts ax and bx, a known difference d between the two amounts fixes x.

(a-b)x = d

For a direct proportion, set the two ratios equal with matching units in matching positions.

a/b = c/d

For inverse proportions, the product of the two matching quantities is the same in every scenario.

x1y1 = x2y2

The new ratio, after changes are applied to the old amounts

(ax ± p)/bx ± q = c/d

New ratio when one part changes by amount p

(ax + p)/bx = c/d

New ratio after both parts change by different amounts

(ax + p)/bx + q = c/d

A percent change to one part of the ratio becomes a concrete addition or subtraction.

(ax ± (p/100)(ax))/bx = c/d

The distance traveled at a constant rate

d = r · t

Average speed for two equal legs at speeds a and b

2ab/(a+b)

Combined time for two workers with solo times a and b

Tcombined = ab/(a+b)

The unknown solo rate is the combined rate minus the known solo rate.

1/b = 1/t - 1/a

Staggered schedules

Wi = Ri × ti, ∑i Wi = 1

Time to complete a remaining fraction f at rate r.

t = f/r

The weighted average is the sum of each group's average times its size, divided by the total size.

Weighted average = (a1n1 + a2n2 + ... + ankn)/(n1 + n2 + ... + nk)

Mixing solutions

c1v1 + c2v2 = c(v1 + v2)

Lines, angles and triangles

The sum of all angles around a point

θ1 + θ2 + ... + θn = 360°

Sum of angle expressions on a line

a + b + c + ... = 180

The sum of the interior angles of a triangle

A + B + C = 180°

The third side of a triangle, given the other two sides

a - b < x < a + b

The area of a triangle is one half times the base times the perpendicular height.

A = 1/2bh

The relationship between the two legs and the hypotenuse of a right triangle

a² + b² = c²

The side ratio for a 45-45-90 triangle, where x is the length of each leg and h is the hypotenuse.

legs : hypotenuse = 1 : 1 : √2
h = x√2

Side lengths of a 30-60-90 triangle, where x is the side opposite the 30-degree angle

short leg = x, long leg = x√3,
hypotenuse = 2x

If two triangles have equal corresponding angles, their corresponding sides are proportional.

a/a' = b/b' = c/c'

For similar triangles, the ratio of areas equals the square of the ratio of corresponding sides.

A1/A2 = (s1/s2)²

A line parallel to one side of a triangle creates a smaller similar triangle.

a/a' = b/b'

Right triangle similarity

h = √pq, a = √pc, b = √qc

Angles on a straight line sum to 180 degrees

α + β = 180°

Valid third side of a triangle with sides a and b

|a - b| < c < a + b

Polygons, circles and solids

Total interior angles of an n-sided polygon

Sum of angles = (n - 2) × 180°

Area of a parallelogram

A = b × h

The area of a trapezoid equals the average of the two parallel bases times the perpendicular height.

area = (b1 + b2)/2 × h

The diagonal of a rectangle from its length and width

d² = l² + w²

Perimeter of a polygon with sides s1, s2, ..., sn

P = s1 + s2 + ... + sn

The circumference and area of a circle in terms of its radius

circumference = 2πr and area = πr²

The radius is half the diameter

r = d/2

Arc length as a fraction of the circumference

arc length = (θ/360) × 2πr

Area of a sector

Asector = (θ/360)πr²

Tangent lines

r² + t² = d²

Half the chord length, the distance from the center to the chord, and the radius form a right triangle.

(c/2)² + d² = r²

Right triangles in circles

a² + b² = (2r)²

Volume of a rectangular solid

V = l × w × h

Volume of a cylinder

V = πr²h

Surface area of a rectangular solid

SA = 2lw + 2lh + 2wh

Total surface area of a cylinder

SA = 2πr² + 2πrh

The space diagonal of a rectangular solid from its length, width, and height

d = √l² + w² + h²

Coordinate geometry

Distance between two points

d = √(x2 - x1)² + (y2 - y1)²

Midpoint of a segment

M = ((x1 + x2)/2, (y1 + y2)/2)

The midpoint of a segment with endpoints (x1, y1) and (x2, y2)

((x1 + x2)/2, (y1 + y2)/2)

Unknown coordinates

√((x2 - x1)² + (y2 - y1)²) = d
((x1 + x2)/2, (y1 + y2)/2) = (mx, my)

The slope-intercept form of a line, where m is the slope and b is the y-intercept

y = mx + b

Slope of a line through two points (x1, y1) and (x2, y2)

m = (y2 - y1)/(x2 - x1)

The y-intercept b given slope m and a point (x1, y1) on the line

b = y1 - m · x1

The x-coordinate where lines y = m1x + b1, and y = m2x + b2 intersect

x = (b2 - b1)/(m1 - m2)

The vertex of a parabola and its direction of opening

y = a(x - h)² + k

X-coordinate of each intersection point

f(x) = g(x)

Perpendicular lines

m1 · m2 = -1

Statistics

The mean is the sum divided by the count, so the sum equals the mean times the count.

mean = sum/count and
sum = mean × count

The missing value, given the target mean and all other values

x = μ · n - S

Mean and median of an ordered set

mean = sum/n, median = middle value of the ordered data

The interval of values within k standard deviations of the mean

mean - k · SD ≤ x ≤ mean + k · SD

Range of a set

R = xmax - xmin

Interquartile range, the spread of the middle half

IQR = Q3 - Q1

The share of data within one standard deviation of the mean in a normal distribution

μ - σ ≤ x ≤ μ + σ
contains about 68% of the data

One-sided percent from the within-range percent

pk = Pk/2

Population standard deviation

σ = √(∑(xi - μ)²/n)

Normal distribution

P(μ - σ < X < μ + σ) ≈ 68%
P(μ - 2σ < X < μ + 2σ) ≈ 95%
P(μ - 3σ < X < μ + 3σ) ≈ 99.7%

Counting, probability, sets and data interpretation

The number of ways to arrange r of n distinct objects in a row

n!/(n - r)!

Arrangements of n items where k specified items must sit together

(n - k + 1)! × k!

Number of unordered groups of r chosen from n

C(n, r) = n!/(r!(n - r)!)

Valid committees when two specific members refuse to serve together

C(n, r) - C(n - 2, r - 2)

Total ways to choose from distinct groups

Total = C(n1, r1) × C(n2, r2) × ... × C(nk, rk)

Selections with at least one of a required type

C(n, r) - C(m, r)

Ordered outcomes from choosing r from n and then arranging them

C(n, r) × r! = P(n, r)

Probability that event A or event B occurs, accounting for overlap

P(A or B) = P(A) + P(B) - P(A and B)

Probability that all independent events occur

P(all) = P(A) × P(B) × ...

Total for two overlapping groups, counting the both-region once

total = A + B - both + neither

Weighted average across table rows

x̄ = (∑(valuei × weighti))/∑weighti

Average of the data rows after any summary row is removed

x̄ = ∑xi/n

Amount for a category, where p is the percent share and T is that pie's total

A = p × T

Format rules

The four fixed outcomes of every quantitative comparison

A > B, A < B, A = B, or D

Digit restrictions

N = n1 × n2 × ... × nk

Number of unordered pairs from n distinct items

(n/2) = n(n - 1)/2

The solution set for a bounded variable is the interval between its minimum and maximum values.

a ≤ x ≤ b

An absolute value inequality of the form |x| > a produces two disjoint intervals.

|x| > a ⇒ x ∈ (-∞, -a) ∪ (a, ∞)

Converting a known range of one variable into the range of a linked variable

a ≤ x ≤ b, y = f(x)
⇒ min{f(a), f(b)} ≤ y ≤ max{f(a), f(b)}